exist, the branching cavity has a loaded Q equal to $\frac{2}{3}Q_L$ and it is not matched. At midband, entering in any port, the normalized reflected power is 1/9 and the normalized transmitted power to each of the other two ports is 4/9.

By definition, the Q of a cavity is

$$Q = \frac{w\omega}{P} \tag{16}$$

where w is the energy stored in the cavity, ω is the angular frequency and P is the power leaving the cavity.

The loaded Q of either cavity when they are operating together, that is, the Q measured in the dropped channel is

$$Q_L = \frac{w\omega}{4|S_{12}|^2 P_i} \tag{17}$$

where P_i is the power carried by either of the two radially propagating waves that build the standing wave in each cavity. The calculation is straightforward and

$$Q_L = \frac{256\pi}{X_1^2 X_2^2} \frac{\zeta_0}{\lambda_g} \frac{a'^2}{\lambda^2} \left(\frac{b}{a'} - 1\right)\left(\frac{a'}{d}\right)^4. \tag{18}$$

The intrinsic Q of a cavity is obtained from (16) when P is the power dissipated in the walls, and the result is

$$Q_i = \frac{2(b - a')^3}{\delta\lambda^2} \frac{1}{1 + \left(\frac{b - a'}{\zeta_0}\right)^3}. \tag{19}$$

$\delta = \sqrt{2/\omega\mu g}$ is the skin depth, ω is the angular frequency, μ the permeability and g the conductivity of the metal.

Eqs. (18) and (19) may be applied to either of the cavities by choosing the appropriate dimensions given in (14) and (15).

Coupled-Mode Description of Crossed-Field Interaction*

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Summary—**The coupled-mode theory is developed for two-dimensional M-type flow, and a system of five coupled-mode equations is obtained. A fifth degree secular equation is found for the perturbed propagation constants of the system. Under weak space-charge field conditions, both the forward-wave and backward-wave interactions may be described in terms of only two coupled modes. The two-mode theory is applied to the calculation of starting conditions for the M-BWO, and to the M-FWA. The conditions for beating-wave amplification are determined, and the variation of the mode amplitudes with distance is given.**

INTRODUCTION

THE coupled-mode theory has been used extensively to describe the operation of the traveling-wave amplifier and backward-wave oscillator.[1–3] In these analyses, the four mode equations refer to the two beam space-charge waves and the two circuit waves. In both the O-type amplifier and the backward-wave oscillator, the best interaction occurs near synchronism between the slow space-charge wave and the circuit wave, and three waves are sufficient to describe the interaction, the fourth being far out of synchronism with the electron beam. Under large space-charge conditions in O-type devices, the interaction is accurately described by utilizing only the slow space-charge wave and the forward circuit wave. The coupled-mode analysis has great utility in obtaining a clear understanding of the detailed interaction mechanism.

Heretofore, the M-type interaction has not been studied with the coupled-mode theory. The general analysis is somewhat more complicated than that for the O-type, since there are five waves involved, which leads to a fifth-degree secular equation. When the RF structure is matched at its output, and the device is operated near synchronism between the electron beam and the foward circuit wave, the interaction is principally due to two waves. This two-wave interaction occurs for low space-charge conditions, unlike the corresponding conditions in the O-type tube. It is the purpose of this paper to develop the coupled-mode description for planar M-type amplifiers and oscillators and to show how this description may be used to investigate growing-wave gain, beating-wave gain and start-oscillation phenomena.

* Received by the PGMTT, November 29, 1960.
† Electron Physics Lab., Dept. of Elec. Engrg., The University of Michigan, Ann Arbor, Mich. This work was supported by the U. S. Signal Corps under Contract No. DA 36-039 sc-78260.

[1] J. R. Pierce, "Coupling of modes of propagation," *J. Appl. Phys.*, vol. 25, pp. 179–183; February, 1954.

[2] J. R. Pierce, "The wave picture of microwave tubes," *Bell Sys. Tech. J.*, vol. 33, pp. 1343–1372; November, 1954.

[3] R. W. Gould, "A coupled mode description of the backward-wave oscillator and the Kompfner dip condition," IRE TRANS. ON ELECTRON DEVICES, vol. ED-2, pp. 37–42; October, 1955.

Circuit and Ballistic Equations

It is convenient to formulate the circuit and ballistic equations using the equivalent circuit analysis, and following the methods of Pierce.[4] The geometry envisaged is that of a relatively thin planar strip beam flowing in orthogonal electrostatic and magnetostatic fields between planar electrodes as illustrated in Fig. 1.

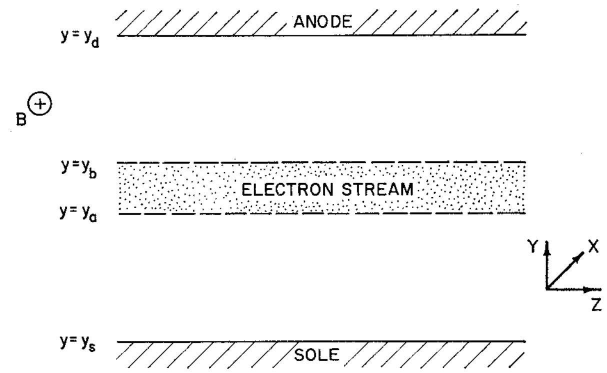

Fig. 1—M-type interaction configuration.

It is assumed that the space-charge conditions satisfy Brillouin flow requirements. It is well known that this is a problem in two-dimensional space-charge flow, and hence, the fields vary in the transverse dimension.

The circuit, ballistic, and continuity equations are expressed as shown below for both the M-FWA and the M-BWO.

$$\frac{\partial V_c}{\partial z} \pm j\beta_0 K_c I_c = 0 \tag{1}$$

$$\frac{\partial I_c}{\partial z} \pm j\beta_0 \frac{1}{K_c} V_c = -\frac{\partial i_b}{\partial z} \tag{2}$$

$$\ddot{y} = \eta\left[\frac{\partial(\Phi V_c)}{\partial y} - B\dot{z}\right] \tag{3}$$

$$\ddot{z} = \eta\left[\frac{\partial(\Phi V_c)}{dz} + B\dot{y}\right] \tag{4}$$

$$\frac{\partial i_b}{\partial z} + \frac{\partial \rho_E}{\partial t} = 0 \tag{5}$$

where

$$i_b = \Phi(y)(\rho u_0 + \rho_0 \dot{z})$$

and

$$\rho_E = \rho\Phi(y) + \rho_0\Phi'(y)y. \tag{6}$$

The quantities introduced in the above equations are defined as follows:

V_c = RF voltage on the circuit
I_c = RF current along the circuit
K_c = circuit impedance
β_0 = unperturbed circuit phase constant
ρ = beam space-charge density
ρ_E = total effective space-charge density
i_b = effective electron beam convection current density
$\Phi(y)$ = transverse potential dependence, and
B = magnetic induction.

In the case of the double signs, the upper one refers to the M-FWA and the lower to the M-BWO. It is convenient to introduce the following new variables into (3) and (4), the ballistics equations:

$$V_{1b} \triangleq -\frac{u_0\dot{z}}{\eta\Phi}$$

$$V_{2b} \triangleq -\frac{u_0\dot{y}}{\eta\Phi}$$

$$\beta_c \triangleq \eta B/u_0. \tag{7}$$

The result is

$$\frac{\partial V_{1b}}{\partial z} + j\beta_e V_{1b} = \beta_c V_{2b} - \frac{\partial V_c}{\partial z} \tag{8}$$

and

$$\frac{\partial V_{2b}}{\partial z} + j\beta_e V_{2b} = -\beta_c V_{1b} - V_c\frac{\Phi'(y)}{\Phi(y)}. \tag{9}$$

Eq. (5), the continuity equation, is written in terms of the new variables, using (6), as

$$i_b - \frac{j}{\beta_e}\frac{\partial i_b}{\partial z} = \frac{2C^3}{K_c}(V_{1b} - j\alpha V_{2b}), \tag{10}$$

where

$$\alpha \triangleq \frac{-\Phi'(y)}{\beta_e\Phi(y)} = \frac{\Phi'(y)}{j\Gamma_0\Phi(y)}$$

$$C^3 \triangleq \frac{\eta\Phi^2 K_c I_0}{2u_0^2}$$

Γ_0 = unperturbed circuit spatial propagation constant for the z-direction,

and

$$\Phi'(y) = \frac{d\Phi(y)}{dy}.$$

A beam impedance K_b is defined by $1/K_b = 2C^3\beta_e/K_c\beta_c$. Eq. (10) is now written as

$$\frac{\partial i_b}{\partial z} + j\beta_e i_b - \frac{\beta_c}{K_b}(jV_{1b} + \alpha V_{2b}) = 0. \tag{11}$$

[4] J. R. Pierce, "Traveling-Wave Tubes," D. Van Nostrand Co., Inc., New York, N. Y., ch. 15; 1950.

Eqs. (8) and (9) are transformed, using (1) and the definitions of α and C:

$$\frac{\partial V_{1b}}{\partial z} + j\beta_e V_{1b} - \beta_c V_{2b} \mp j\beta_0 K_c I_c = 0 \qquad (12)$$

and

$$\frac{\partial V_{2b}}{\partial z} + j\beta_e V_{2b} + \beta_c V_{1b} \pm \alpha\beta_0 K_c I_c = 0. \qquad (13)$$

Mode Formalism

The coupled-mode formalism is now conveniently introduced into the circuit, ballistics and continuity equations with the aid of the following definitions. These definitions are made to simplify the final equations, and they give the wave amplitudes as being proportional to square root power.

Circuit Modes:

$$P_c \triangleq \frac{1}{2}\left(\frac{V_c}{\sqrt{K_c}} + \sqrt{K_c}\, I_c\right) \qquad (14)$$

$$Q_c \triangleq \frac{1}{2}\left(\frac{V_c}{\sqrt{K_c}} - \sqrt{K_c}\, I_c\right). \qquad (15)$$

Beam Modes:

$$P_{1b} \triangleq \frac{1}{2\sqrt{K_b}}(V_{1b} + jV_{2b}) \qquad (16)$$

$$P_{2b} \triangleq \frac{1}{2\sqrt{K_b}}(V_{1b} - jV_{2b}) \qquad (17)$$

$$Q_b = \frac{1}{2}\left(\frac{\sqrt{\alpha}\, V_{1b}}{\sqrt{K_b}} - j\frac{V_{2b}}{\sqrt{\alpha K_b}} \quad \sqrt{\frac{K_b}{\alpha}}\, i_b\right). \qquad (18)$$

After introduction of the definitions contained in (14)–(18), the circuit and beam coupled-mode equations are expressed as follows:

$$\frac{\partial P_{1b}}{\partial z} + j(\beta_e + \beta_c)P_{1b} \mp j\frac{(1-\alpha)}{2}\beta_0\left(\frac{K_c}{K_b}\right)^{1/2}(P_c - Q_c) = 0 \qquad (19)$$

$$\frac{\partial P_{2b}}{\partial z} + j(\beta_e - \beta_c)P_{2b} \mp j\frac{(1+\alpha)}{2}\beta_0\left(\frac{K_c}{K_b}\right)^{1/2}(P_c - Q_c) = 0 \qquad (20)$$

$$\frac{\partial Q_b}{\partial z} + j\beta_e Q_b \mp j\beta_0\left(\frac{\alpha K_c}{K_b}\right)^{1/2}(P_c - Q_c) = 0 \qquad (21)$$

$$\frac{\partial P_c}{\partial z} \pm j\beta_0 P_c + j\frac{(1-\alpha)}{2}\left(\frac{K_c}{K_b}\right)^{1/2}(\beta_e + \beta_c)P_{1b} - j\frac{(1+\alpha)}{2}\cdot\left(\frac{K_c}{K_b}\right)^{1/2}(\beta_e - \beta_c)P_{2b} + j\beta_e\left(\frac{\alpha K_c}{K_b}\right)^{1/2}Q_b = 0 \qquad (22)$$

and

$$\frac{\partial Q_c}{\partial z} \mp j\beta_0 Q_c - j\frac{(1-\alpha)}{2}\left(\frac{K_c}{K_b}\right)^{1/2}(\beta_e + \beta_c)P_{1b} + j\frac{(1+\alpha)}{2}\cdot\left(\frac{K_c}{K_b}\right)^{1/2}(\beta_e - \beta_c)P_{2b} - j\beta_e\left(\frac{\alpha K_c}{K_b}\right)^{1/2}Q_b = 0. \qquad (23)$$

Eqs. (19)–(23) constitute the five coupled-mode equations for M-type interactions. These equations are written in terms of the unperturbed wave propagation constants; the next step is to develop the secular equation from their simultaneous solution. The uncoupled phase constants will be perturbed as the coupling between the individual modes is increased.

The roots of the determinant formed from the coefficients of the modes P_c, Q_c, Q_b, P_{1b}, and P_{2b} give the following secular equation. It is assumed that all modes vary as $\exp(j\omega t - \Gamma z)$.

$$[(j\beta_e - \Gamma)^2 + \beta_c^2](-\Gamma + j\beta_0)(\Gamma^2 + \beta_0^2) = \beta_e\beta_0\Gamma^2 H^2\left((j\beta_e - \Gamma) + 2j\beta_c\frac{\alpha}{1+\alpha^2}\right), \qquad (24)$$

where

$$H^2 \triangleq 2C^3(1 + \alpha^2).$$

This dispersion equation agrees with that obtained by Pierce.[4] It is worthwhile to proceed one step further and verify the possibility of amplification in such a system. Under the condition that $\beta_e + \beta_c \approx \beta_0$, approximately only the modes P_{1b} and P_c are coupled (cyclotron-wave interaction); hence, the determinant becomes

$$\begin{vmatrix} -\Gamma + j(\beta_e + \beta_c) & -j\dfrac{(1-\alpha)}{2}\beta_0\left(\dfrac{K_c}{K_b}\right)^{1/2} \\ j\left(\dfrac{K_c}{K_b}\right)^{1/2}\dfrac{(1-\alpha)}{2}(\beta_e + \beta_c) & -\Gamma + j\beta_0 \end{vmatrix} = 0. \qquad (25)$$

For cyclotron-wave interaction, the electron motion is essentially circular, with equal y-directed and z-directed energies, and the modes Q_c, Q_b, and P_{2b} are excited to a negligible extent due primarily to weak coupling to the circuit. Expanding the determinant and defining $\Gamma = j\beta_0(1+p)$ yields

$$p = \pm\frac{j}{2}\frac{(1-\alpha)}{(1+\alpha^2)^{1/2}}\left(\frac{\beta_e}{\beta_c}\right)^{1/2}H, \qquad (26)$$

which agrees with Pierce. Gain occurs for all values of α, and is a maximum for $\alpha = -1$.

Forward-Wave Amplifier

Under low space-charge conditions, the M-FWA may be described in terms of two coupled modes, P_c and P_{1b}. This is counter to the O-type FWA, which may be described in terms of two coupled modes only for high space-charge conditions. It is convenient to write the solution as

$$P_c = Ae^{-\Gamma_1 z} + Be^{-\Gamma_2 z}, \quad (27)$$

and then P_{1b} may be expressed as

$$P_{1b} = -(j\beta_0 - \Gamma_2)Ae^{-\Gamma_1 z} - (j\beta_0 - \Gamma_1)Be^{-\Gamma_2 z}. \quad (28)$$

The boundary conditions for the forward-wave amplifier are expressed as follows for an initially unmodulated stream:

$$P_c = 1$$
$$P_{1b} = 0 \quad \text{at} \quad z = 0. \quad (29)$$

It is convenient to assume approximate synchronism between the slow space-charge wave and the forward circuit wave ($\beta_e \approx \beta_0$), and write the propagation constants in the following form:

$$\Gamma_{1,2} = j\beta_e + \beta_e D\delta_{1,2}.$$

Under the above condition for maximum excitation of the near-synchronous waves, the cyclotron waves are negligibly excited, and hence, their effect is neglected. Using the above form for Γ_n yields

$$A = \frac{1}{1 - \delta_2/\delta_1} \quad \text{and} \quad B = \frac{1}{1 - \delta_1/\delta_2};$$

then

$$P_c = e^{-j\beta_e}\left[\frac{\delta_1}{\delta_1 - \delta_2}e^{-\theta\delta_1} - \frac{\delta_2}{\delta_1 - \delta_2}e^{-\theta\delta_2}\right], \quad (30)$$

where

$$\theta \triangleq \beta_e D z = 2\pi D N_s.$$

For low space-charge conditions, the solution of the dispersion equation for the perturbed propagation constants yields

$$\delta_{1,2} = \pm\sqrt{1 - (b/2)^2} + j\frac{b}{2}, \quad (31)$$

where $b \triangleq (\beta_0 - \beta_e)/\beta_e D$.

For exact synchronism, the δ's are purely real and each wave is excited in equal amplitude, which indicates that the initial loss factor for the M-FWA is -6.02 db. It should be noted that the mode amplitudes are proportional to square-root power and hence, proportional to the RF voltage in a matched system. It is interesting to examine the first term of (31). For

$$1 - (b/2)^2 > 0; \quad \text{growing waves}$$
$$= 0; \quad \text{transition region}$$
$$< 0; \quad \text{beating waves.}$$

The transition point occurs at $b = 2$; for larger values of b, both δ's are purely imaginary.

The growing-wave regime has been investigated, and excellent agreement has been obtained with the results of Muller.[5] The square of the mode amplitude in the beating-wave regime is readily expressed as

$$|P_c|^2 = A^2 + B^2 + 2AB\cos\theta(y_1 - y_2), \quad (32)$$

where y_1, y_2 are the imaginary parts of the propagation constants. At the input $\theta = 0$,

$$|P_c| = A + B = 1, \quad (33)$$

where B is a *negative* number. Under these conditions, there are two purely propagating waves on the circuit which are out of phase at the input and beat together, adding in phase at the output. Since $A > 0$ and $B < 0$, the maximum of (32) occurs at (in phase condition)

$$\theta = \frac{\pi}{(y_1 - y_2)} = \frac{\pi}{2\sqrt{(b/2)^2 - 1}}. \quad (34)$$

At the maximum point,

$$|P_c| = A - B. \quad (35)$$

It is seen that the position of maximum gain, and the maximum gain are solely functions of b. The gain and optimum length are given as a function of b in Table I.

The predictions made using the two-wave coupled-mode theory agree exactly with the general M-FWA theory. The gain as a function of θ and b is shown in Fig. 2.

TABLE I
LENGTH AND GAIN FOR BEATING-WAVE INTERACTION

b	θ_{opt}	DN	Voltage Gain	G_{db}
2.02	11.11	1.77	7.15	17.10
2.05	7.05	1.12	4.6	13.25
2.1	4.97	0.79	3.32	10.40
2.2	3.45	0.55	2.41	7.60
2.3	2.78	0.44	2.04	6.16

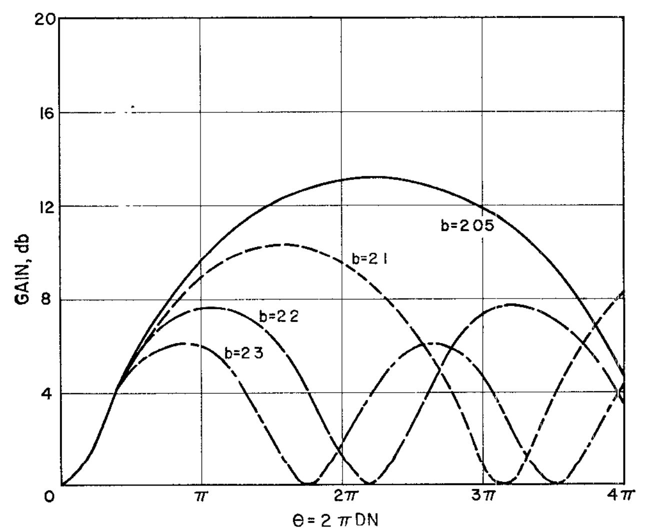

Fig. 2—Gain vs θ for zero space charge.

[5] M. Muller, "Traveling-wave amplifiers and backward-wave oscillators," PROC. IRE, vol. 42, pp. 1651–1659; November, 1954.

Backward-Wave Oscillator

The M-type backward-wave oscillator may also be described in terms of two coupled modes for low space-charge conditions. In this instance, we deal with the backward-circuit wave and the slow space-charge wave. For the condition that $\beta_e \approx \beta_0$, the coupled modes are Q_c and Q_b, and the mode equations are

$$\frac{\partial Q_c}{\partial z} + j\beta_0 Q_c - j\beta_e\left(\frac{\alpha K_c}{K_b}\right)^{1/2} Q_b = 0 \tag{36}$$

and

$$\frac{\partial Q_b}{\partial z} + j\beta_e Q_b - j\beta_0\left(\frac{\alpha K_c}{K_b}\right)^{1/2} Q_c = 0. \tag{37}$$

The roots of the determinant of the coefficients of the above system give for the coupled propagation constants

$$\Gamma_{1,2} = \frac{j}{2}\left[(\beta_0 + \beta_e) \pm \sqrt{(\beta_0 - \beta_e)^2 + 4\beta_e\beta_0\alpha\frac{K_c}{K_b}}\right]. \tag{38}$$

Here again, it is convenient to write

$$Q_c = A_1 e^{-\Gamma_1 z} + B_1 e^{-\Gamma_2 z}, \tag{39}$$

and then

$$Q_b = (j\beta_0 - \Gamma_1)A_1 e^{-\Gamma_1 z} + (j\beta_0 - \Gamma_2)B_1 e^{-\Gamma_2 z}, \tag{40}$$

where A_1 and B_1 are yet undetermined. In the case of the backward-wave oscillator, it is desired to find the point along the structure $z>0$ where the energy in the circuit field has been completely transferred to a beam space-charge field. These boundary conditions on Q_c and Q_b are

$$Q_b = 0 \quad \text{at} \quad z = 0$$
$$Q_c = 0 \quad \text{at} \quad z = L. \tag{41}$$

The above conditions give

$$\frac{A_1}{B_1} = -\frac{j\beta_0 - \Gamma_2}{j\beta_0 - \Gamma_1} = -e^{-\Gamma_2 L}/e^{-\Gamma_1 L}. \tag{42}$$

Upon introduction of the propagation constants and the velocity parameter, (42) becomes

$$\frac{b + \sqrt{b^2 + \frac{4\alpha}{D_i^2}\frac{K_c}{K_b}}}{b - \sqrt{b^2 + \frac{4\alpha}{D_i^2}\frac{K_c}{K_b}}} = \exp\left\{j\theta\sqrt{b^2 + \frac{4\alpha}{D_i^2}\frac{K_c}{K_b}}\right\}. \tag{43}$$

The radicals in (43) may be considerably simplified by using the definitions of interaction parameter and transverse field variation suggested by Dombrowski:[6]

$$D_i^2 = \frac{\omega}{\omega_c}\frac{K}{2}\frac{I_0}{V_0}G^2$$

$$G^2 = -\frac{1}{2}\frac{\sinh 2j\Gamma_0(\bar{y} - y_s)}{\sinh^2[j\Gamma_0(y_d - y_s)]}. \tag{44}$$

The result is

$$\frac{b + \sqrt{b^2 + 4}}{b - \sqrt{b^2 + 4}} = e^{j\theta\sqrt{b^2+4}}. \tag{45}$$

Eq. (45) is satisfied, and an oscillation condition exists for

$$b = 0$$

$$D_i N_s = \frac{\theta}{2\pi} = 0.25. \tag{46}$$

These are exactly the start-oscillation conditions for a M-BWO when the space-charge fields are weak as compared to the circuit fields, independent of the beam position between the anode and sole plates. This result indicates that the energy is completely transferred between the beam and the circuit every 90 degrees.

Conclusions

The general interaction between wave and beam for two-dimensional M-type flow has been described in terms of coupled modes, resulting in five mode equations. They are written in terms of the uncoupled mode propagation constants and arise out of the ballistic, circuit, and continuity equations. Under the conditions of weak space-charge fields, both the forward-wave and the backward-wave interactions may be described satisfactorily in terms of two coupled modes. In the forward-wave case, the mode equations were used to calculate the optimum length and gain of a beating-wave amplifier; for backward-wave interaction, starting conditions were determined.

The development of a satisfactory coupled-mode theory for a device hinges on the separate determination of the wave coupling parameter in terms of device parameters. Based on the above results, it is felt worthwhile to apply this coupled-mode approach to a study of cyclotron-wave interaction in crossed fields, in order to obtain a physical picture of the interaction phenomena. This technique could also be applied to a study of cyclotron-wave parametric devices.

[6] G. E. Dombrowski, "A Small-Signal Theory of Electron-Wave Interaction in Crossed Electric and Magnetic Fields," Electron Tube Lab., The University of Michigan, Ann Arbor, Mich. Tech. Rept. No. 22; October, 1957.